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Dynamics of flexible slender cylinders in axial flow

Part 1. Theory

By M. P. PAIDOUSSIS
Chalk River Nuclear Laboratories, Atomic Energy of Canada Limited

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A general theory is presented to account for the small, free, lateral motions of a flexible, slender, cylindrical body immersed in fluid flowing parallel to the position of rest of its axis. The cylinder is either clamped or pinned at both ends, or clamped at the upstream end and free at the other; it lies in a horizontal plane wherein all motion is considered to be confined. It is shown that for sufficiently large flow velocities the cylinder may be subject to buckling and oscillatory instabilities in its first and higher flexural modes, respectively. It is shown that for cylinders with both ends supported the oscillatory instabilities are specifically caused by lateral frictional forces, and that in the absence of hydrodynamic-drag effects only buckling is possible. The same applies for cylinders supported at the upstream end and with a very long, gradually tapering free end. The critical conditions of stability, expressed in dimensionless form, are evaluated extensively for clamped-free and pinned-pinned cylinders, illustrating the effect of the various system parameters on stability.

1. Introduction

In the last decade the study of dynamics of flexible tubes containing flowing fluid has revealed a number of interesting phenomena. Niordson (1953) and Handelman (1955), dealing with tubes supported at both ends, reported that for sufficiently large flow velocities such systems buckle like columns under axial loading; this is usually referred to as buckling instability. Benjamin (1961), dealing with the general dynamical problem of articulated pipe systems conveying fluid, showed that, when such systems possess one free end, for sufficiently high flow velocities unstable oscillations may develop. Later Paidoussis (1963) demonstrated the existence of unstable oscillations of continuously flexible tubular cantilevers conveying fluid.

In this study we are concerned with the dynamics of flexible slender cylinders surrounded by, rather than containing, flowing fluid. Provided that the flow direction coincides with the axis of the cylinder at rest, then, for small motions about the position of rest, the forces exerted by the fluid in the two cases of external and internal flow are closely similar. This becomes evident on considering that the forces exerted by the fluid, excepting those due to fluid friction, in both cases arise from lateral acceleration of the flowing fluid caused by lateral motion of the cylinder. In external flow, this acceleration is suffered by the virtual or 'associated' mass of fluid (Munk 1924), which is dynamically equivalent to the contained mass of fluid in internal flow. Hawthorne (1961), taking advantage of this similarity, investigated the stability of flexible tubes towed in water and demonstrated that buckling instability is possible in such systems.

The theoretical and experimental work reported here and in part 2 is intended as the first stage of a study into the mechanism of fluid-induced vibration of fissile-fuel elements in liquid-cooled nuclear-reactor channels. The work in this paper aims at determining the effect of steady flow on small lateral motions of cylindrical beams immersed in fluid flowing parallel to their axis, under various end constraints. Particular attention is focused on identifying the various possible hydroelastic instabilities and on establishing criteria for stability, especially for cylinders pinned at both ends and cylinders clamped at the upstream end and free at the other.

2. General theory

Equation of small motions

The system under consideration consists of a flexible cylindrical body of circular cross-section, immersed in an incompressible fluid of density ρ flowing with uniform velocity U parallel to the x-axis, which coincides with the position of rest of the cylinder axis. The cylinder is considered to be either fixed at the upstream end and free at the other, as in figure 1, or supported at both ends. Except for a short tapering piece by which the cylinder is terminated at a free end (which will be considered with the boundary conditions), the cylinder has uniform cross-sectional area S, mass per unit length m and flexural rigidity EI. The x- and y-axes lie in a horizontal plane wherein all motions of the cylinder are supposed to be confined.

We consider small lateral motions of the cylinder about its position of rest, during which the angle of incidence i and $\partial i/\partial x$ remain sufficiently small so that (*a*) no separation occurs in cross-flow, and (*b*) the fluid forces on each element of the cylinder may be assumed to be the same as those acting on a corresponding element of a long straight cylinder of the same cross-sectional area and inclination. Furthermore, the fluid is supposed to be contained by boundaries sufficiently distant from the cylinder to have negligible influence on its motion. We suppose that the cylinder is given a small lateral displacement $y(x,t)$ from the straight position. The resultant relative velocity between the cylinder and the fluid flowing past it (Lighthill 1960) is given by

$$v(x,t) = \frac{\partial y}{\partial t} + U\frac{\partial y}{\partial x}. \tag{1}$$

Near the cylinder this lateral flow is identical with the two-dimensional potential flow that would result from the motion of the cylinder through fluid at rest with velocity $v(x,t)$. We suppose that this flow has momentum Mv per unit length of cylinder, where M is the virtual mass of the fluid per unit length which is equal to ρS for a circular cylinder, provided that the wavelength of motion is large in comparison with the diameter of the cylinder (Niordson 1953). The rate of change of this momentum per unit length is $([\partial/\partial t] + U[\partial/\partial x])\,(Mv)$ and gives rise to an equal and opposite lateral force on the cylinder (Lighthill 1960).

We now consider a small element δx of the cylinder, as shown in figure 2. Denoting the axial tension by T, and the viscous forces per unit length in the normal and longitudinal directions by F_N and F_L, respectively, the equations of motion in the two directions may be written as follows:

$$\frac{\partial T}{\partial x} + F_L = 0, \tag{2}$$

and
$$\frac{\partial Q}{\partial x} - F_N - M\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 y - m\frac{\partial^2 y}{\partial t^2} + \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right) = 0. \tag{3}$$

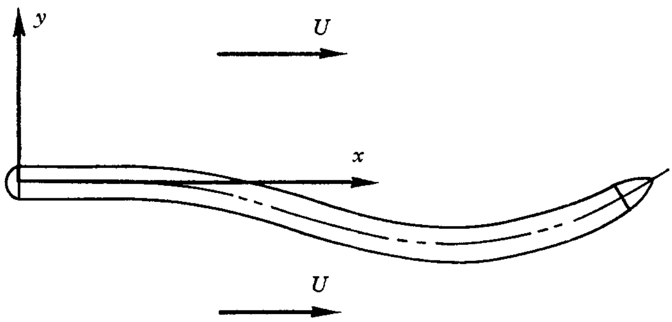

FIGURE 1. Diagram of a clamped-free cylinder in axial flow.

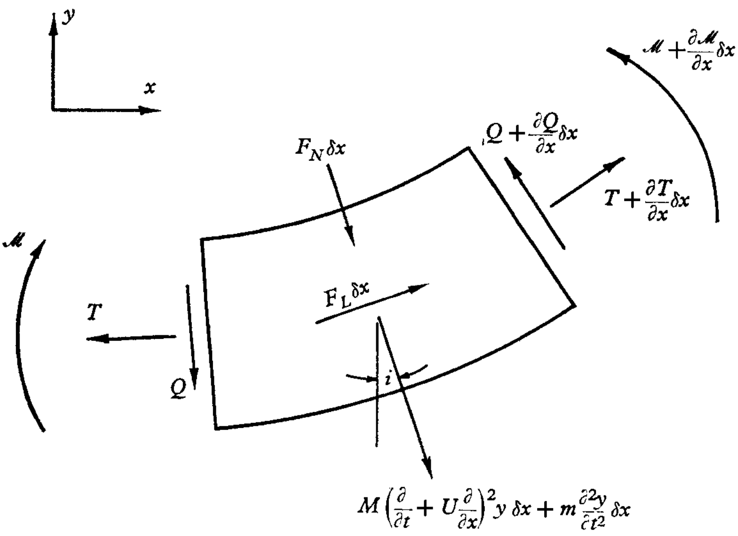

FIGURE 2. Forces and moments acting on an element of the cylinder.

For small lateral motions of large wavelength, inertial forces in the axial direction are of second order of magnitude and have been neglected. Similarly, the effects of angular acceleration of the element are neglected and the lateral shear force Q is, by elementary beam theory,

$$Q = -\frac{\partial \mathscr{M}}{\partial x} = -EI\frac{\partial^3 y}{\partial x^3}. \tag{4}$$

The viscous forces acting on long inclined cylinders have been discussed by Taylor (1952). For turbulent boundary layers Taylor noted that viscous forces will depend on the exact nature of roughness. Where it may be assumed that roughness consists of a number of projections pointing equally in all directions, he proposed that

$$F_N = \tfrac{1}{2}\rho D U^2 (C_{Dp} \sin^2 i + C_f \sin i)$$

and

$$F_L = \tfrac{1}{2}\rho D U^2 C_f \cos i,$$

where D is the cylinder diameter, and C_{Dp} and C_f are the coefficients associated with form and friction drag for a cylinder in cross-flow.

For the motions considered here $\sin i \ll 1$ and, as only linear terms will be considered in the analysis, it is assumed that the viscous forces may be adequately represented by

$$F_N = \tfrac{1}{2}(M/D)\, U^2 c_N \sin i \quad \text{and} \quad F_L = \tfrac{1}{2}(M/D)\, U^2 c_T, \tag{5}$$

in which ρ was eliminated by the use of $M = \frac{1}{4}\pi D^2 \rho$. In general, c_N and c_T are not equal. The angle i may be related to the normal and axial components of flow velocity by

$$i = \sin^{-1}(v/U), \tag{6}$$

which substituted into (5) along with (1) yields

$$F_N = \tfrac{1}{2}c_N (M/D)\, U\left(\frac{\partial y}{\partial t} + U\frac{\partial y}{\partial x}\right) \quad \text{and} \quad F_L = \tfrac{1}{2}(M/D)\, U^2 c_T. \tag{7}$$

The axial tension may be found by substituting F_L from (7) into (2) and integrating, assuming the upstream support always to coincide with $x = 0$. Where the downstream end is free, integration from x to $x = L$ yields

$$T(x) = T(L) + \tfrac{1}{2}c_T M U^2 (L-x)/D.$$

A non-zero value of $T(L)$ can only arise from form drag at the free end, which may be considered proportional to $\frac{1}{2}\rho U^2 S$. Accordingly, we write $T(L) = \frac{1}{2}c'_T M U^2$, where c'_T is the form drag coefficient and $T(x) = \frac{1}{2}c_T M U^2(L-x)/D + \frac{1}{2}c'_T M U^2$. Where both ends are supported, we assume that the downstream support may be moved axially, before the flow is turned on, to impose on the cylinder a tension T_0. No further motion of the support is allowed, however, so that the overall extension from $x = 0$ to L caused by fluid friction is zero; this gives a force distribution along the length equal to $\frac{1}{2}c_T M U^2(\frac{1}{2}L - x)/D$. Hence, by superposition we obtain in this case $T(x) = T_0 + \frac{1}{2}c_T M U^2(\frac{1}{2}L - x)/D$. In general, therefore, we may write

$$T(x) = \gamma T_0 + \tfrac{1}{2}c_T \frac{MU^2}{D}\{(1-\tfrac{1}{2}\gamma)\, L - x\} + \tfrac{1}{2}(1-\gamma)\, c'_T M U^2, \tag{8}$$

in which $\gamma = 0$ if the downstream end is not supported, and $\gamma = 1$ if it is.

Substituting now (4), (7) and (8) into (3), we obtain the equation of small lateral motions

$$EI\frac{\partial^4 y}{\partial x^4} + M\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 y - \frac{\partial}{\partial x}\left(\tfrac{1}{2}c_T \frac{MU^2}{D}\{(1-\tfrac{1}{2}\gamma)\, L - x\}\frac{\partial y}{\partial x}\right)$$

$$-\{\gamma T_0 + \tfrac{1}{2}(1-\gamma)\, c'_T M U^2\}\frac{\partial^2 y}{\partial x^2} + \tfrac{1}{2}c_N \frac{MU}{D}\left(\frac{\partial y}{\partial t} + U\frac{\partial y}{\partial x}\right) + m\frac{\partial^2 y}{\partial t^2} = 0. \tag{9}$$

It is noted that in this equation no account has been taken of damping forces in the material of the cylinder.

Boundary conditions

When both ends of the cylinder are supported, then the boundary conditions applying to a beam must be satisfied. Accordingly, at a pinned end we must have $y = \partial^2 y/\partial x^2 = 0$, and at a clamped end $y = \partial y/\partial x = 0$.

If the downstream end is free, it is assumed that the cross-sectional area tapers smoothly from S to zero in a distance sufficiently short, so that y and the lateral velocity v may be considered constant. This requirement allows the forces acting on the tapered end to be lumped and considered in appropriate boundary conditions (Hawthorne 1961). Equating the lateral shear and inertial forces to the rate of change of lateral momentum over the tapered end, say for $L-l < x < L$, we obtain

$$\int_{L-l}^{L} \frac{\partial Q}{\partial x}\,dx - f\int_{L-l}^{L}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)[M(x)\,v]\,dx - \int_{L-l}^{L} m(x)\,\frac{\partial^2 y}{\partial t^2}\,dx = 0. \qquad (10)$$

The parameter f, which is equal to unity for slender-body, inviscid-flow theory, has been introduced because the theoretical lateral force at the free end may not be fully realized as a result of (a) the lateral flow not being truly two-dimensional, since the fluid has opportunity to pass around rather than over the tapered end (Munk 1924), and (b) boundary-layer effects (Hawthorne 1961). Accordingly, f will normally be less than unity. [Munk (1924) showed that the momentum of the lateral (inviscid) flow about an inclined rigid body of revolution is proportional to $K_y - K_x$, where K_y and K_x are respectively the lateral and axial virtual masses of the fluid for the whole body. For ellipsoids of revolution Lamb (1932) calculated that $K_x/K_y = 1$ for $L/D = 1$, and $K_x/K_y = 0{\cdot}022$ for $L/D = 9{\cdot}47$. Thus for elongated bodies K_x may be neglected in comparison with K_y. For the flexible body under consideration, however, we are dealing with distributed virtual masses along the length. Over the cylindrical portion of the body the axial virtual mass is, of course, identically zero, and the lateral virtual mass per unit length is M. For the tapered end-piece, on the other hand, the lumped, lateral virtual mass K_{ye} which equals $\int_{L-l}^{L} M(x)\,dx$ is not necessarily much greater than the corresponding axial virtual mass K_{xe} since l/D is not always large. Accordingly, it is clear that f, which may now be assumed proportional to $1-K_{xe}/K_{ye}$ may be considerably less than unity, quite apart from boundary-layer effects.]

We now write $M(x) = \rho S(x)$ and $m(x) = \rho_b S(x)$ and, since y and v are considered constant over the tapered end, (10) leads to

$$\int_{L-l}^{L} \frac{\partial Q}{\partial x}\,dx - f\rho U v\int_{L-l}^{L} \frac{\partial S}{\partial x}\,dx - (\rho_b + f\rho)\,\frac{\partial^2 y}{\partial t^2}\int_{L-l}^{L} S(x)\,dx = 0,$$

which yields $$EI\,\frac{\partial^3 y}{\partial x^3} + fMU\left(\frac{\partial y}{\partial t}+U\frac{\partial y}{\partial x}\right) - (m+fM)\,x_e\,\frac{\partial^2 y}{\partial t^2} = 0,$$

where $$x_e = \frac{1}{S}\int_{L-l}^{L} S(x)\,dx.$$

Furthermore, it may be assumed that there is no bending moment at the free end, so that

$$\partial^2 y/\partial x^2 = 0.$$

It is understood, of course, that $l/L \ll 1$ so that the boundary conditions may be considered to apply at $x = L$.

Dimensionless parameters

Before proceeding with the analysis, it is desirable to express the problem in dimensionless terms and accordingly we put

$$\left.\begin{aligned} \xi = x/L, \quad \eta = y/L, \quad \tau = \{EI/(m+M)\}^{\frac{1}{2}} t/L^2, \\ \beta = M/(m+M), \quad \Gamma = T_0 L^2/EI, \quad \epsilon = L/D \quad \text{and} \quad u = (M/EI)^{\frac{1}{2}} UL. \end{aligned}\right\} \tag{11}$$

Substituting into (9), we obtain

$$\frac{\partial^4 \eta}{\partial \xi^4} + \{u^2[1 - \tfrac{1}{2}\epsilon c_T(1 - \tfrac{1}{2}\gamma - \xi) - \tfrac{1}{2}(1-\gamma) c_T'] - \gamma\Gamma\} \frac{\partial^2 \eta}{\partial \xi^2} + 2\beta^{\frac{1}{2}} u \frac{\partial^2 \eta}{\partial \xi \partial \tau} + \tfrac{1}{2}\epsilon(c_N + c_T) u^2 \frac{\partial \eta}{\partial \xi} + \tfrac{1}{2}\epsilon c_N \beta^{\frac{1}{2}} u \frac{\partial \eta}{\partial \tau} + \frac{\partial^2 \eta}{\partial \tau^2} = 0. \tag{12}$$

In dimensionless form, the boundary conditions for a body pinned at both ends are

$$\eta = \partial^2\eta/\partial\xi^2 = 0 \quad \text{at} \quad \xi = 0 \quad \text{and} \quad \xi = 1, \tag{13}$$

and for a body clamped at the upstream end and free at the other

$$\left.\begin{aligned} \eta = \partial\eta/\partial\xi = 0 \quad \text{at} \quad \xi = 0, \\ \text{and} \quad \frac{\partial^2\eta}{\partial\xi^2} = \frac{\partial^3\eta}{\partial\xi^3} + fu^2\frac{\partial\eta}{\partial\xi} + f\beta^{\frac{1}{2}}u\frac{\partial\eta}{\partial\tau} - \{1+(f-1)\beta\}\chi\frac{\partial^2\eta}{\partial\tau^2} = 0 \quad \text{at} \quad \xi = 1, \end{aligned}\right\} \tag{14}$$

where $\chi = x_e/L$.

3. Analysis

Let us consider motions of the cylinder of the form

$$\eta = Y(\xi)\, e^{i\omega\tau}, \tag{15}$$

where ω is a dimensionless frequency defined by $\omega = \{(M+m)/EI\}^{\frac{1}{2}}\,\Omega L^2$, in which Ω is the circular frequency of motion. In general Ω will be complex and the system will be stable or unstable accordingly as the imaginary part of ω is positive or negative; in the case of neutral stability Im $(\omega) = 0$. Substituting (15) into (12), we obtain

$$\frac{d^4Y}{d\xi^4} + \{u^2[1 - \tfrac{1}{2}\epsilon c_T(1 - \tfrac{1}{2}\gamma - \xi) - \tfrac{1}{2}(1-\gamma)c_T'] - \gamma\Gamma\}\frac{d^2Y}{d\xi^2} + \{\tfrac{1}{2}\epsilon(c_N + c_T)u^2 + 2\beta^{\frac{1}{2}}u\omega i\}\frac{dY}{d\xi} + \{-\omega^2 + \tfrac{1}{2}\epsilon c_N\beta^{\frac{1}{2}}u\omega i\}\,Y = 0. \tag{16}$$

The system under consideration has an infinite number of degrees of freedom. The complete solution of the dynamical problem therefore involves the determination of the infinite set of frequencies of the normal modes of oscillation of the system, as continuous functions of u, β, ϵc_N, ϵc_T, c'_T, f, etc.

Two methods of solution are presented. In the first, the deflexion of the cylinder in the flow is expressed by superposition of the infinite set of eigenfunctions of a cylinder in still fluid. This method is useful in cases where the boundary conditions in flow and in still fluid are identical, i.e. when the cylinder is supported at both ends. For clamped-free cylinders, on the other hand, the solution is found by the second method, where the deflexion of the cylinder is expressed as a power series in ξ.

Solution by beam eigenfunction series

Let $Y(\xi)$ be given by

$$Y(\xi) = \sum_{r=1}^{\infty} a_r \phi_r(\xi), \tag{17}$$

where the functions

$$\phi_r(\xi) = A\cos\lambda_r\xi + B\sin\lambda_r\xi + C\cosh\lambda_r\xi + D\sinh\lambda_r\xi$$

	Pinned-pinned cylinders	Clamped-clamped cylinders
$b_{rs}(r \neq s)$	$\dfrac{2\lambda_r\lambda_s}{\lambda_r^2-\lambda_s^2}\{(-1)^{r+s}+1\}$	$\dfrac{4\lambda_r^2\lambda_s^2}{\lambda_r^4-\lambda_s^4}\{(-1)^{r+s}-1\}$
b_{rr}	0	0
$c_{rs}(r \neq s)$	0	$\dfrac{4\lambda_r^2\lambda_s^2}{\lambda_r^4-\lambda_s^4}(\lambda_r\sigma_r-\lambda_s\sigma_s)\{(-1)^{r+s}+1\}$
c_{rr}	$-\lambda_r^2$	$\lambda_r\sigma_r(4-\lambda_r\sigma_r)$
$d_{rs}(r \neq s)$	$\dfrac{4\lambda_r^3\lambda_s}{(\lambda_r^2-\lambda_s^2)^2}\{1-(-1)^{r+s}\}$	$\dfrac{4\lambda_r^2\lambda_s^2(\lambda_r\sigma_r-\lambda_s\sigma_s)}{\lambda_r^4-\lambda_s^4}(-1)^{r+s}-\dfrac{3\lambda_r^4+\lambda_s^4}{\lambda_r^4-\lambda_s^4}b_{rs}$
d_{rr}	$-\frac{1}{2}\lambda_r^2$	$\frac{1}{2}\lambda_r\sigma_r(4-\lambda_r\sigma_r)$

TABLE 1. The constants b_{rs}, c_{rs} and d_{rs}.

are the beam eigenfunctions (Bishop & Johnson 1960), λ_r are the eigenvalues and A, B, C, D are constants determined by the boundary conditions. For a pinned-pinned beam, $A = C = D = 0$ and $B = 1$, while, for a clamped-free beam, $A = -1$, $B = \sigma_r$, $C = 1$ and $D = -\sigma_r$, where

$$\sigma_r = (\cosh\lambda_r - \cos\lambda_r)/(\sinh\lambda_r - \sin\lambda_r).$$

Substituting (17) into (16) and recalling that $\gamma = 1$ for cylinders supported at both ends, we obtain

$$\sum_{r=1}^{\infty}[\lambda_r^4\phi_r + \{u^2(1-\tfrac{1}{4}\epsilon c_T)-\Gamma\}\phi_r'' + \tfrac{1}{2}\epsilon c_T u^2\xi\phi_r'' + \{\tfrac{1}{2}\epsilon(c_N+c_T)u^2 + 2\beta^{\frac{1}{2}}u\omega i\}\phi_r' + \{-\omega^2 + \tfrac{1}{2}\epsilon c_N\beta^{\frac{1}{2}}u\omega i\}\phi_r]a_r = 0. \tag{18}$$

We now define

$$\phi_r' = \sum_{s=1}^{\infty} b_{rs}\phi_s, \quad \phi_r'' = \sum_{s=1}^{\infty} c_{rs}\phi_s, \quad \text{and} \quad \xi\phi_r'' = \sum_{s=1}^{\infty} d_{rs}\phi_s, \tag{19}$$

which substituted into (18) yield

$$\sum_{r=1}^{\infty}\left\{(\lambda_r^4+A)\,\phi_r+B\sum_{s=1}^{\infty}b_{rs}\phi_s+C\sum_{s=1}^{\infty}c_{rs}\phi_s+D\sum_{s=1}^{\infty}d_{rs}\phi_s\right\}a_r=0, \tag{20}$$

where $A=-\omega^2+\frac{1}{2}\epsilon c_N\beta^{\frac{1}{2}}u\omega i$, $B=\frac{1}{2}\epsilon(c_N+c_T)\,u^2+2\beta^{\frac{1}{2}}u\omega i$, $C=(1-\frac{1}{4}\epsilon c_T)\,u^2-\Gamma$ and $D=\frac{1}{2}\epsilon c_T u^2$ were introduced for convenience.

The b_{rs}, c_{rs} and d_{rs} are determined from the properties of the eigenfunctions and the boundary conditions by methods similar to those discussed by Bishop & Johnson (1960). Their values, in terms of λ_i and σ_i, are given in table 1 for cylinders with pinned and clamped ends.

Expanding (20), we now obtain an infinite number of linear equations:

$$\left.\begin{aligned}&(\lambda_1^4+A+Bb_{11}+Cc_{11}+Dd_{11})\,a_1+(Bb_{21}+Cc_{21}+Dd_{21})\,a_2\\&\qquad\qquad+(Bb_{31}+Cc_{31}+Dd_{31})\,a_3+\ldots=0,\\&(Bb_{12}+Cc_{12}+Dd_{12})\,a_1+(\lambda_2^4+A+Bb_{22}+Cc_{22}+Dd_{22})\,a_2\\&\qquad\qquad+(Bb_{32}+Cc_{32}+Dd_{32})\,a_3+\ldots=0,\\&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{etc.}\end{aligned}\right\} \tag{21}$$

Since these equations are linear in a_r, the condition for non-trivial solution is that the determinant of the coefficients of a_r must vanish. This determinant is of infinite order, but, assuming that the system may be adequately described by synthesis of the n lowest eigenfunctions, it reduces to one of nth order. Thus an implicit expression may be obtained for ω as a function of u, β, ϵc_N, ϵc_T and Γ. The complexity of the determinantal equation, however, renders the explicit formulation of this functional relationship impracticable. On the other hand, for particular values of u, β, ϵc_N, ϵc_T and Γ, clearly we can evaluate the complex frequency of any of the normal modes of the system by numerical methods.

Solution by power series

For convenience (16) is written in the form

$$\frac{d^4Y}{d\xi^4}+a\frac{d^2Y}{d\xi^2}+b\xi\frac{d^2Y}{d\xi^2}+c\frac{dY}{d\xi}+eY=0, \tag{22}$$

and, after substituting (15) into (14), the boundary conditions are

$$\left.\begin{aligned}&Y(0)=\left(\frac{dY}{d\xi}\right)_{\xi=0}=0.\\ \text{and}\qquad&\left(\frac{d^2Y}{d\xi^2}\right)_{\xi=1}=\left(\frac{d^3Y}{d\xi^3}+h\frac{dY}{d\xi}+jY\right)_{\xi=1}=0,\end{aligned}\right\} \tag{23}$$

where $a=u^2(1-\frac{1}{2}\epsilon c_T-\frac{1}{2}c'_T)$, since $\gamma=0$,

$$b=\tfrac{1}{2}\epsilon c_T u^2,\quad c=\tfrac{1}{2}\epsilon(c_N+c_T)\,u^2+2\beta^{\frac{1}{2}}u\omega i,$$

$$e=-\omega^2+\tfrac{1}{2}\epsilon c_N\beta^{\frac{1}{2}}u\omega i,\quad h=fu^2\quad\text{and}\quad j=f\beta^{\frac{1}{2}}u\omega i+\{1+(f-1)\,\beta\}\,\chi\omega^2.$$

Let us try the solution

$$Y(\xi)=\sum_{r=0}^{\infty}A_r\xi^r, \tag{24}$$

where A_r are constants to be determined. It is assumed that the motion of the cylinder may be described adequately by a conveniently small number of terms, in which case the series may be truncated at a finite value of r, say at $r = n$.

Substituting (24) into (23), we obtain $A_0 = A_1 = 0$ and

$$\left.\begin{aligned} 2A_2 + 6A_3 + \sum_{r=4}^{n} r(r-1)A_r &= 0,\\ (2h+j)A_2 + (6+3h+j)A_3 + \sum_{r=4}^{n}[r(r-1)(r-2)+rh+j]A_r &= 0. \end{aligned}\right\} \quad (25)$$

Substituting (24) into (22) and collecting terms, we obtain

$$\left.\begin{aligned} 24A_4 + 2aA_2 &= 0,\\ 120A_5 + 6A_3 + 2(b+c)A_2 &= 0,\\ 360A_6 + 12aA_4 + (6b+3c)A_3 + eA_4 &= 0,\\ &\cdots\cdots\cdots\\ n(n-1)(n-2)(n-3)A_n + a(n-2)(n-3)A_{n-2} &\\ + b(n-3)(n-4)A_{n-3} + c(n-3)A_{n-3} + eA_{n-4} &= 0. \end{aligned}\right\} \quad (26)$$

From the first three equations it is evident that all the A_r may be expressed as linear combinations of A_2 and A_3 alone. Consequently, (25) are linear equations in A_2 and A_3, and the determinant of their coefficients must vanish for non-trivial solution. Hence, by this method also we obtain an implicit relation between ω and $u, \beta, \epsilon c_N$, etc.

4. The frequency as a function of the flow velocity

We suppose that for any given physical system the parameters β, ϵ, c_N, c_T and, where applicable, Γ or f and c'_T can be determined and that they are all independent of the flow velocity. Using representative values of these parameters, the complex frequency of the three lower modes of a number of systems is calculated, starting with $u = 0$ and increasing the flow velocity in small steps. The results demonstrate the general character of the dynamical behaviour of the system for varying u, illustrating some of the modes of instability to which it may be subjected.

Computational method

When $u = 0$, the allowable values of ω are wholly real numbers, in the absence of damping forces. For a cylinder pinned at both ends, we have $\omega = \pi^2$, $4\pi^2$, $9\pi^2, \ldots,$ etc. when $\Gamma = 0$, which values relate to the first, second and higher modes of a pinned-pinned beam. For a cylinder clamped at one end and free at the other, the corresponding frequencies are those of a cantilever, i.e. $\omega = 3{\cdot}516$, $22{\cdot}034$, $61{\cdot}697$, etc., approximately, modified by the departure from cylindrical geometry at the tapered free end.

It is noted that both methods of solution lead to a vanishing determinant with complex terms. Accordingly, for a given value of u, Re (ω) and Im (ω) were determined by systematic trial and error (see Gregory & Paidoussis 1966), such that the real and imaginary parts of the determinant went to zero simultaneously. The calculations were done on the Bendix G-20 computer of the Chalk River Nuclear Laboratories. The computer calculated the complex frequency for each mode of a given system for increasing values of u, starting with $u = 0$ for which ω is known at least approximately. The computation was repeated with an increasing number of terms in the series solution to ensure that the results converge to the desired accuracy.

Pinned-pinned cylinder

The complex frequency of the three lowest modes of a system with $\beta = 0{\cdot}10$, $\epsilon c_N = \epsilon c_T = 1$ and $\Gamma = 0$ is displayed as an Argand diagram in figure 3. It is noted that small flow velocities act to damp free oscillations of the system. As

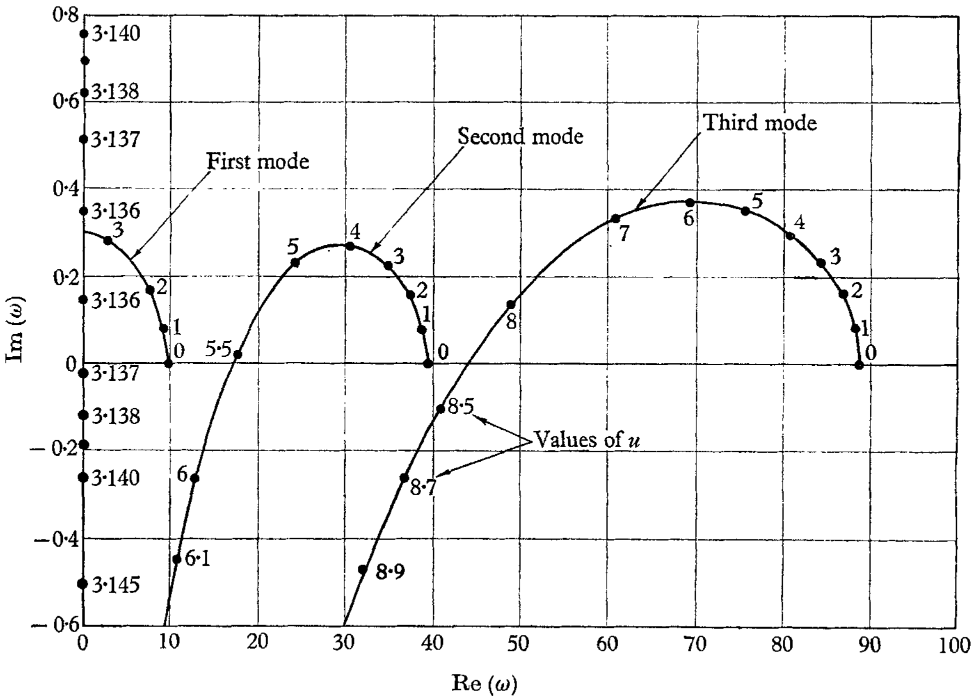

FIGURE 3. The dimensionless complex frequency of the three lowest modes of a pinned-pinned cylinder ($\beta = 0{\cdot}1$, $\epsilon c_N = \epsilon c_T = 1$, $\Gamma = 0$) as a function of the dimensionless flow velocity u.

the flow velocity increases, however, the system may become unstable in all three modes. The locus of the first mode bifurcates on the [Im (ω)]-axis, one branch receding from the origin and the other approaching and eventually crossing it; this evidently indicates buckling instability ($\omega = 0$). The instabilities associated with the second and third mode, on the other hand, which occur at higher flow velocities than buckling, are oscillatory.

The existence of a buckling instability is not unexpected in view of the close dynamical similarity between this system and those investigated by Niordson (1953) and Benjamin (1961), in which the flow is internal. This is not the case for oscillatory instabilities, which have been reported in systems with internal flow only when they possess a free end (Benjamin 1961; Gregory & Paidoussis 1966).

In all cases, Re (ω) and Im (ω) converged to three significant figures by synthesis of the lowest six or seven beam eigenfunctions.

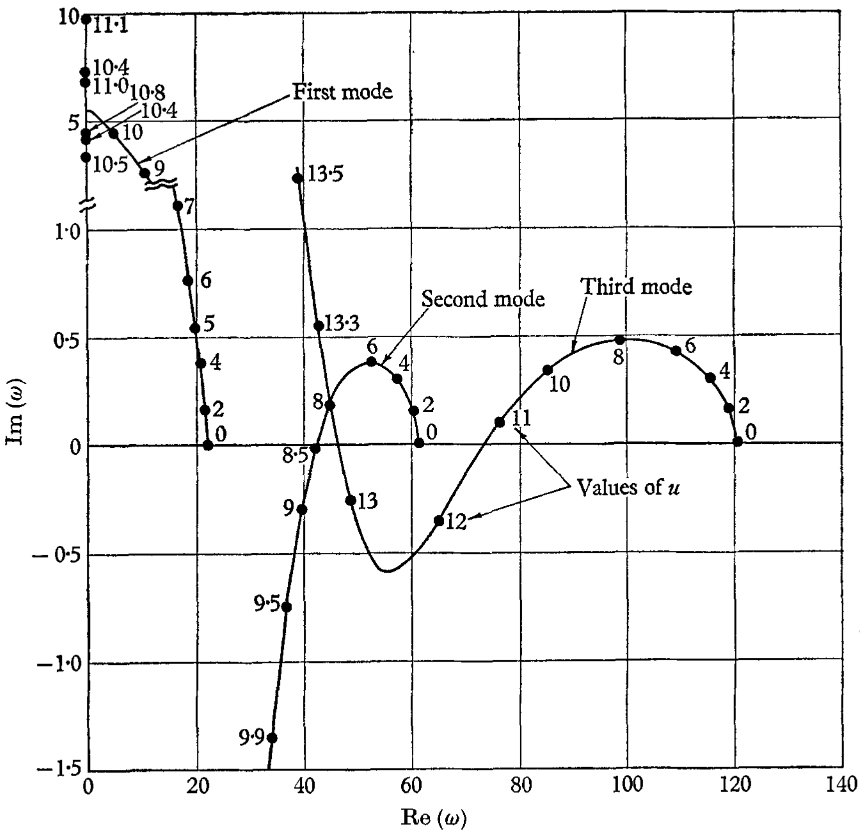

FIGURE 4. The dimensionless complex frequency of the three lowest modes of a clamped-clamped cylinder ($\beta = 0{\cdot}1$, $\epsilon c_N = \epsilon c_T = 1$, $\Gamma = 0$) as a function of the dimensionless flow velocity u.

Clamped-clamped cylinder

The complex frequency of the same cylinder but with the ends clamped is shown in figure 4, exhibiting basically the same behaviour for varying u as when the ends are pinned. In this case, however, buckling does not occur and the second and third mode instabilities occur at higher flow velocities. One of the loci corresponding to the first mode begins to approach the origin, but then, at $u \approx 10{\cdot}5$, it begins to recede further into the stable region ($u = 11{\cdot}1$, Im (ω) $= 9{\cdot}8$). Another notable feature of this system is that the third mode goes back to the stable region after it first crosses over to the unstable one. The behaviour of this system in its first and third modes is not a feature of clamped-clamped systems exclusively; for various combinations of the system parameters, similar behaviour was observed in all three modes of pinned-pinned and clamped-free cylinders.

Clamped-free cylinder

The complex frequency of a system with the upstream end clamped and the other free is shown in figure 5. In this case also the loci of all three modes eventually cross the [Re (ω)]-axis into the unstable region. It is interesting to note that buckling instability is possible in this case, while this phenomenon does not occur

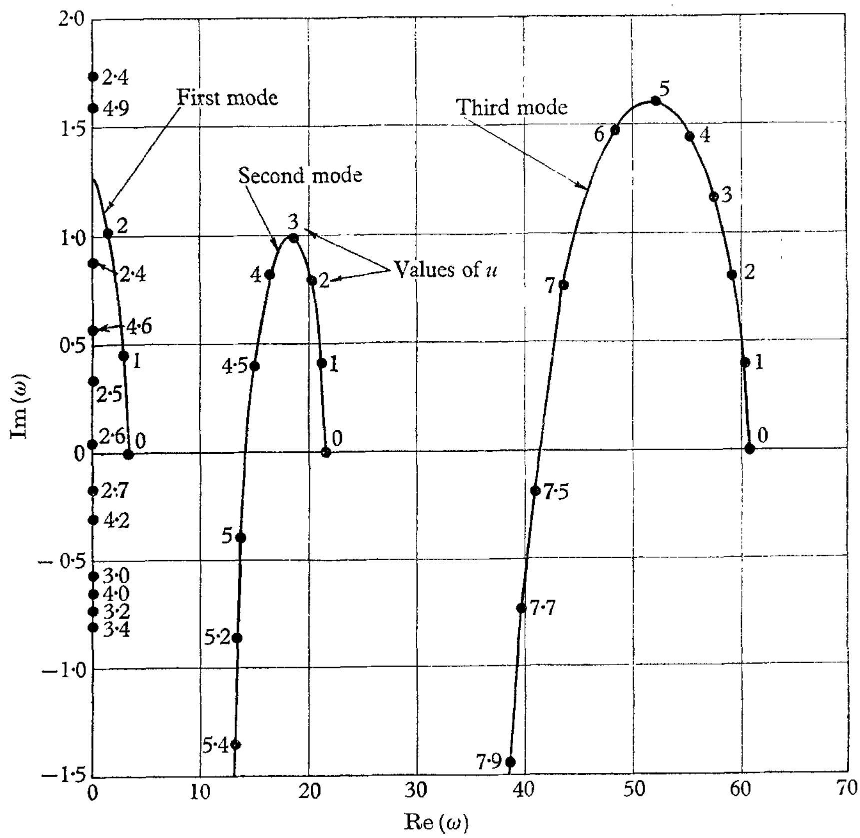

FIGURE 5. The dimensionless complex frequency of the three lowest modes of a clamped-free cylinder ($\beta = 0{\cdot}5$, $\epsilon c_N = \epsilon c_T = 1$, $c_T' = 0$, $f = 0{\cdot}8$, $\chi = 0{\cdot}01$) as a function of the dimensionless flow velocity u.

when the flow is internal (Paidoussis 1963), unless gravity is operative (Benjamin 1961). After the system becomes unstable in its first mode, further increase in the flow velocity eventually causes it to regain stability (the locus re-crosses the origin from negative to positive Im (ω)). This occurs at a flow velocity smaller than that required for second mode instability.

In this case, convergence to three significant figures required up to 50 terms of the power series.

5. The mechanism of instability

The mechanism underlying buckling instability may be illuminated by considering the static equilibrium of the cylinder with both ends supported, which is assumed to have momentarily taken an arbitrary bowed shape $y(x)$. Eliminating the time-dependent terms in (9), we obtain

$$EI\frac{d^4y}{dx^4}+MU^2\frac{d^2y}{dx^2}-T_0\frac{d^2y}{dx^2}-\frac{1}{2}\frac{MU^2}{D}c_T\left(\tfrac{1}{2}L-x\right)\frac{d^2y}{dx^2}+\tfrac{1}{2}(c_N+c_T)\frac{MU^2}{D}\frac{dy}{dx}=0. \qquad (27)$$

If the forces arising from fluid viscosity are neglected ($c_N = c_T = 0$), this equation reduces to the corresponding one for a pipe containing flowing fluid (Gregory & Paidoussis 1966). It is evident, therefore, that for sufficiently high fluid velocity the forces represented by the second term may overcome the flexural and tensile restoring forces, resulting in a monotonic increase in the amplitude of $y(x)$. Actually, the mechanism of instability is more complex than suggested above, particularly if the cylinder has a free end, when the system is inherently non-conservative (cf. Benjamin 1961). Evidence of this will be given later, when it will be seen that form drag at the free end may have a destabilizing effect on the system.

For oscillatory instabilities, the condition of neutral stability is one of dynamic equilibrium where, in the course of one cycle of oscillation, the energy transfer from fluid to cylinder and vice versa exactly balance. When the former exceeds the latter, since the fluid stream may be regarded as a source of infinite energy, the amplitude increases without limit; in the opposite case oscillations are damped.

Benjamin (1961) considered the mechanism of energy transfer in the related problem of fluid flowing in an articulated pipe system; he found that in motion over a time 0 to t_1 which concludes with the system in its original state, if the downstream end is free, the energy gained by the pipes is

$$\Delta W = -\int_0^{t_1} M_i U(\dot{\mathbf{R}}^2 + U\boldsymbol{\tau}.\dot{\mathbf{R}})\,dt, \qquad (28)$$

where M_i is the mass per unit length of the contained fluid and $\boldsymbol{\tau}$ and $\mathbf{R}$ are the tangential and position vectors at the end of the last pipe. It is evident that when U is small the first term in the integrand predominates over the second, and vibrations are damped ($\Delta W < 0$). For sufficiently high U, however, if $\boldsymbol{\tau}$ and $\dot{\mathbf{R}}$ are sufficiently out of phase to give a negative average value to $\boldsymbol{\tau}.\dot{\mathbf{R}}$, it is possible for ΔW to become positive, in which case vibrations are amplified. Thus, for amplified vibrations, for the greater part of a cycle the last pipe must slope backwards to the motion of its free end, i.e. a 'dragging' motion must obtain. By looking at the problem in terms of the Lagrangian or Hamiltonian methods, Benjamin was able to show that ΔW is the rate of working by the *non-conservative* part of the hydrodynamic forces and that this interpretation is not limited to periodic motions.

Another important corollary of (28) is that, when the downstream end of the pipe system is also supported, we have $\Delta W = 0$. This means that oscillations in this case can be neither damped nor amplified by the action of the flow. Alternatively we can say that the hydrodynamic forces are then wholly of *conservative* type.

As remarked previously, the system under consideration here where the flow is external is closely similar to the one discussed above where the flow is internal, as evidenced by the similarity in the equations of motion in the two cases (cf. Gregory & Paidoussis 1966). There are important differences, however. Thus frictional forces are operative in the case of external flow; also, at a free end the shear is generally not zero. Considering (9) we find that the rate of work done on the cylinder in the course of free periodic motions may generally be written as

$$\frac{dW}{dt} = -\int_0^L EI\dot{y}y^{\mathrm{iv}}\,dx - \int_0^L \dot{y}M\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 y\,dx - \tfrac{1}{2}c_N\int_0^L \frac{MU}{D}\dot{y}(\dot{y} + Uy')\,dx,$$

and over one period of oscillation t_1 the work done ΔW is found to be

$$\Delta W = -\int_0^{t_1}\left[\dot{y}\{EIy''' + MU(\dot{y} + Uy')\}\right]_0^L dt - \tfrac{1}{2}c_N\int_0^{t_1}\int_0^L \frac{MU}{D}(\dot{y}^2 + Uy'\dot{y})\,dx\,dt.$$

Applying the boundary conditions for cylinders supported only at the upstream end, we obtain

$$\Delta W_1 = -(1-f)\int_0^{t_1} MU(\dot{y}^2 + Uy'\dot{y})_L\,dt - \tfrac{1}{2}c_N\int_0^{t_1}\int_0^L \frac{MU}{D}(\dot{y}^2 + Uy'\dot{y})\,dx\,dt, \quad (29)$$

and for cylinders supported at both ends

$$\Delta W_2 = -\tfrac{1}{2}c_N\int_0^{t_1}\int_0^L \frac{MU}{D}(\dot{y}^2 + Uy'\dot{y})\,dx\,dt. \quad (30)$$

Cylinders supported at both ends

From (30) it is evident that in the absence of hydrodynamic-drag effects we have $\Delta W_2 = 0$. By analogy to the internal flow case discussed above we can say that, excluding the frictional forces, the hydrodynamic forces are purely of conservative type. In terms of the complex frequency diagrams of § 4, in the absence of frictional forces the frequency of all the modes follows the [Re (ω)]-axis with increasing u, toward the origin; at the origin the locus bifurcates and the two branches continue along the positive and the negative [Im (ω)]-axis. Thus we come to the important conclusion that oscillatory instabilities are entirely due to the effect of frictional forces, in the absence of which the only form of instability possible is buckling. More generally, all departures from the [Re (ω)]-axis in any particular mode, except where Re $(\omega) = 0$, are entirely due to the effect of friction. In fact, Im (ω) is linearly dependent on the value of ϵc_N for small values of u; thus, for $u = 2$ in the example of figure 3, we find that Im $(\omega) = 0{\cdot}155$ when $\epsilon c_N = 1$, and Im $(\omega) = 0{\cdot}310$ when $\epsilon c_N = 2$.

Moreover, it appears that the stability conditions are virtually independent of the value of ϵc_N, provided it is reasonably small (see figure 7). The onset of amplified oscillations evidently corresponds to a change in the character of the mode at a specified flow velocity as determined by the *conservative* forces in the system; then frictional effects, however small, cause amplification of the oscillation, whereas before the change occurred they caused damping. (This change in the character of the mode is essentially like the change from class B to class A behaviour discussed in general terms by Benjamin (1963).)

Inspecting (30) more closely, we observe that the condition of neutral stability ($\Delta W_2 = 0$) requires that

$$\int_0^L \overline{\dot{y}^2}\, dx = -U \int_0^L \overline{\dot{y}y'}\, dx. \tag{31}$$

We note that, for the right-hand side of (31) to be non-zero, different parts of the body must vibrate in quadrature, and this is an effect that can only be produced by the action of flow. (In the absence of flow, the normal modes for both ends pinned are of the form $y = A \sin(\omega t + \phi) \sin(n\pi x/L)$, *where* ϕ *is independent of* x, and the r.h.s. of (31) is zero.) However, if the solution is represented as the sum of a standing and a travelling wave, it is easy to see that the r.h.s. of (31) depends only on the amplitude of the travelling-wave component; furthermore, the phase velocity of this component must be in the direction of flow for the r.h.s. of (31) to be positive. This character of amplified oscillation is corroborated by the experiments of part 2 (Paidoussis 1966).

Cylinders with only the upstream end supported

From (29) we note that energy transfer between the cylinder and the fluid stream may be caused both by inviscid and by viscous hydrodynamic forces. The similarity between (28) and the first term of (29) is striking. One important difference is that the first term of (29) vanishes when $f = 1$, and oscillatory instabilities in this case, as for cylinders with both ends supported, occur (see figure 10) because of frictional forces. This is analogous to Lighthill's (1960) findings, that a fish with a gradually tapering tail and no tail-fin (which corresponds to $f = 1$) cannot produce a net propulsive force; i.e. it cannot swim. For $f < 1$, however, departures from the [Re (ω)]-axis when $U \neq 0$ and oscillatory instabilities can occur quite independently of frictional forces. Thus, it is no accident that the values of Im (ω) are so much larger in figure 5 than in figures 3 and 4 for comparable values of u. In fact, for $c_N = c_T = c'_T = 0$ we find that Im (ω) is almost proportional to $1 - f$ for small u, as shown in table 2 ($\beta = 0{\cdot}2$, $u = 1$).

f	Complex frequency
1	20·98
0·8	$21{\cdot}08 + 0{\cdot}155i$
0·5	$21{\cdot}22 + 0{\cdot}398i$
0	$21{\cdot}41 + 0{\cdot}813i$

TABLE 2

If the frictional forces are considered to play a secondary role to the inviscid hydrodynamic forces in determining the condition of neutral stability, by similarity to the case of internal flow we may expect the same 'dragging' type of motion of the free end during amplified oscillations. This was seen to be the case in the experiments of part 2.

6. The conditions of stability

The results shown in figures 3–5 establish the existence of instabilities induced by flow. In this section, the critical flow velocities for neutral stability and corresponding frequencies associated with these instabilities are calculated systematically, demonstrating the effect of the various system parameters β, ϵc_N, ϵc_T, Γ, etc., on the stability of the system. In view of the large number of these parameters and of the similarities noted in the behaviour of the system under different conditions of end constraint, extensive calculations are confined to pinned-pinned cylinders and clamped-free cylinders. Furthermore, since instability appears to occur first in the first and second modes, the conditions of stability associated with these two modes only are considered.

For a given set of system parameters and for each mode, the values of u and $\mathrm{Re}\,(\omega)$ at the point of neutral stability, where $\mathrm{Im}\,(\omega) = 0$, were determined essentially by the method given in § 4 for determining $\mathrm{Re}\,(\omega)$ and $\mathrm{Im}\,(\omega)$ for given values of u. For buckling $\mathrm{Re}\,(\omega) = 0$ also, and the calculation is further simplified; moreover, the problem in this case is independent of β and χ.

Pinned-pinned cylinders

The dimensionless critical flow velocity for buckling u_{cb} is shown in figure 6. It may be seen that externally applied tension ($\Gamma > 0$) stabilizes the system, and compression ($\Gamma < 0$) destabilizes it. For $\Gamma = -\pi^2$, buckling occurs at zero flow velocity, which coincides with Euler's (1933) result for buckling of long slender columns with pinned ends.

If $c_T = \frac{1}{2}c_N$, increasing ϵ or c_N and c_T stabilizes the system, and for sufficiently large values of ϵc_N no buckling occurs (e.g. $\epsilon c_N = 2\epsilon c_T \geqslant 9{\cdot}8$, $\Gamma = 16$). If $c_T = c_N$, on the other hand, the system is less stable. Distributed longitudinal drag evidently has a destabilizing effect on buckling. Since the downstream support cannot slide axially, longitudinal drag tends to compress the downstream half of the cylinder and increase the tendency to buckle.

The dimensionless critical flow velocity u_{co} and the corresponding frequency ω_{co} for second mode instability are shown in figures 7 and 8. We see that, provided $\epsilon c_N < 1$, it has virtually no effect on the conditions of stability as discussed in § 5. The effect of the various system parameters on stability may be summarized as follows: (*a*) externally applied tension stabilizes and compression destabilizes the system; (*b*) increasing both ϵc_N and ϵc_T usually stabilizes the system; (*c*) if $c_T = c_N$ the system is more stable than if $c_T < c_N$.

Clamped-free cylinders

Figures 9–11 show that the stability of clamped-free cylinders depends strongly on the parameter f. Evidently, as the geometry of the free end departs from an ideally slender shape the system is stabilized quite effectively. In some cases, as for $\epsilon c_N = \epsilon c_T = 1{\cdot}25$, $c'_T = 0$, no buckling occurs at all if $f \leqslant 0{\cdot}8$; furthermore, in this example, if $\beta > 0{\cdot}86$ no second mode instability is possible either. As used in this paper, f virtually controls the boundary conditions at the free end

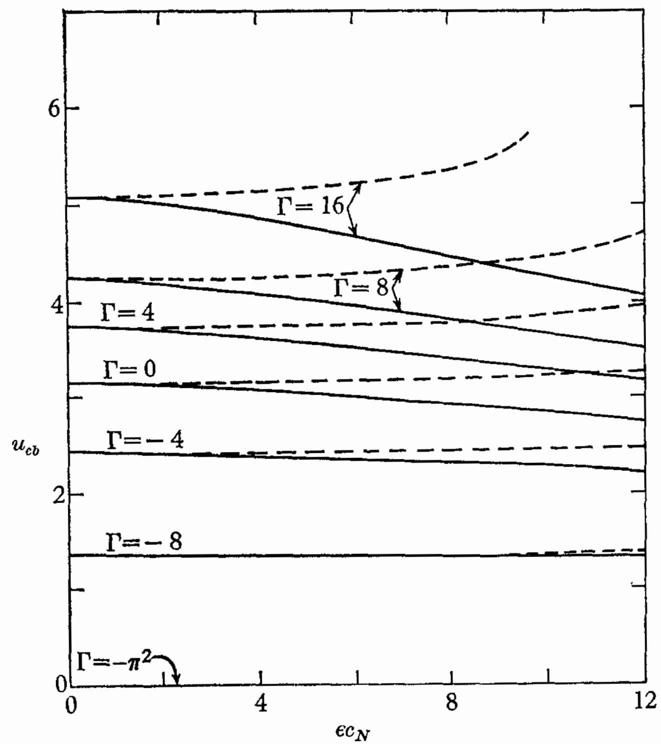

FIGURE 6. The dimensionless critical flow velocity for buckling of pinned-pinned cylinders. ———, $c_T/c_N = 1$; - - - -, $c_T/c_N = \frac{1}{2}$.

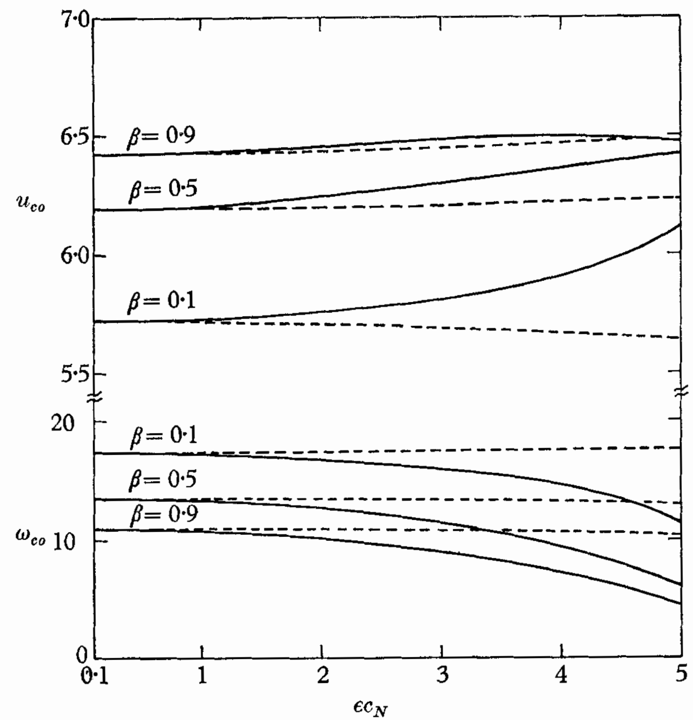

FIGURE 7. The dimensionless critical flow velocity and frequency for second-mode unstable oscillation of pinned-pinned cylinders ($\Gamma = 0$), showing the effect of ϵc_N and ϵc_T on stability. ———, $c_T/c_N = 1$; - - -, $c_T/c_N = \frac{1}{2}$.

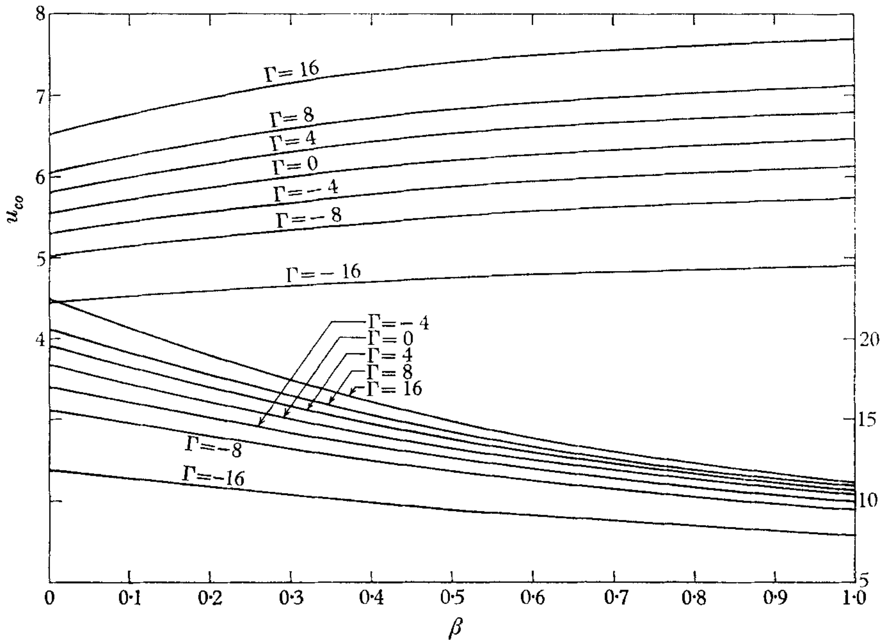

FIGURE 8. The dimensionless critical flow velocity and frequency for second-mode unstable oscillation of pinned-pinned cylinders ($\epsilon c_N = \epsilon c_T = 1$) showing the effect of β and Γ on stability.

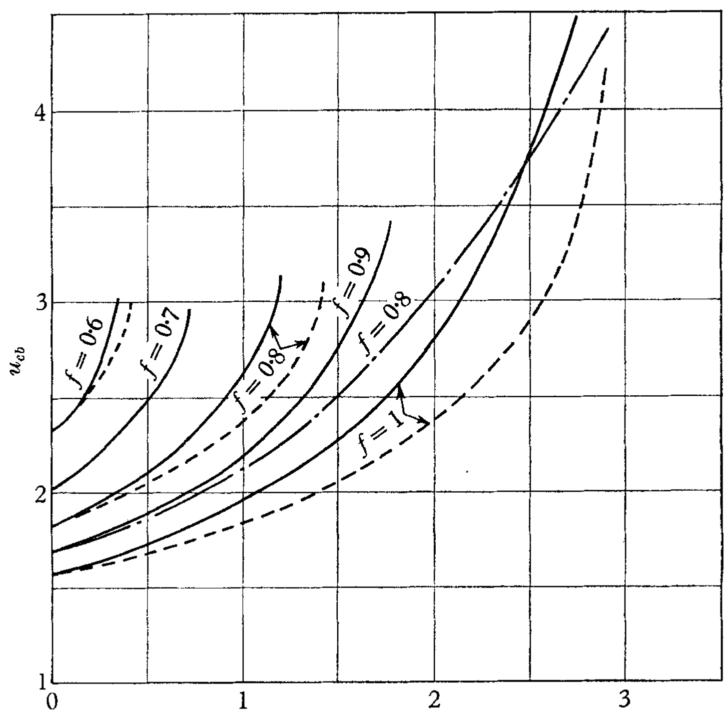

FIGURE 9. The dimensionless critical flow velocity for buckling of clamped-free cylinders. ———, $c_T/c_N = 1$, $c'_T = 0$; —·—·—, $c_T/c_N = 1$, $c'_T = 1$; - - -, $c_T/c_N = \frac{1}{2}$, $c'_T = 0$.

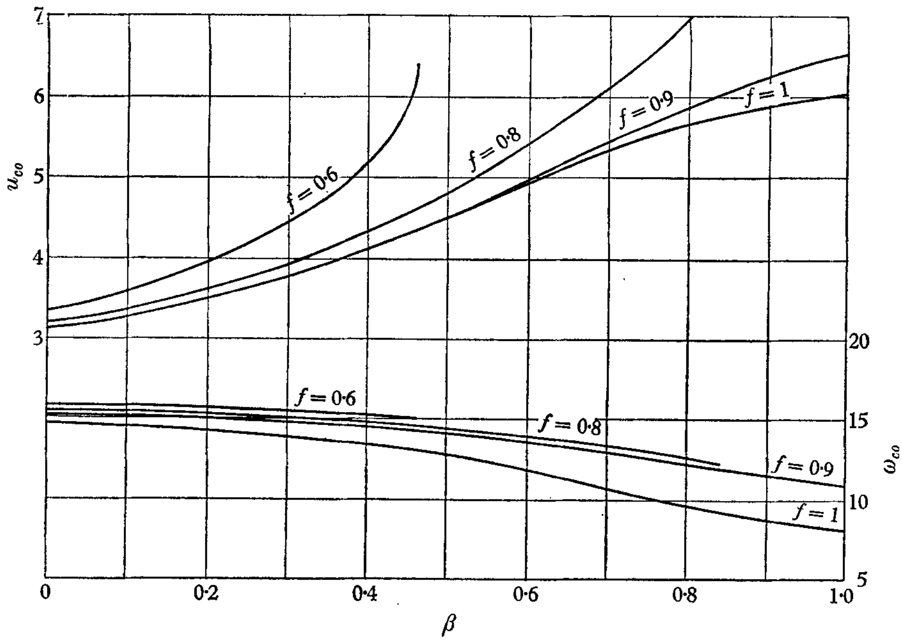

FIGURE 10. The dimensionless critical flow velocity and frequency for second-mode unstable oscillation of clamped-free cylinders ($\epsilon c_N = \epsilon c_T = 1$, $c_T' = 0$, $\chi = 0{\cdot}01$), showing the effects of f and β on stability.

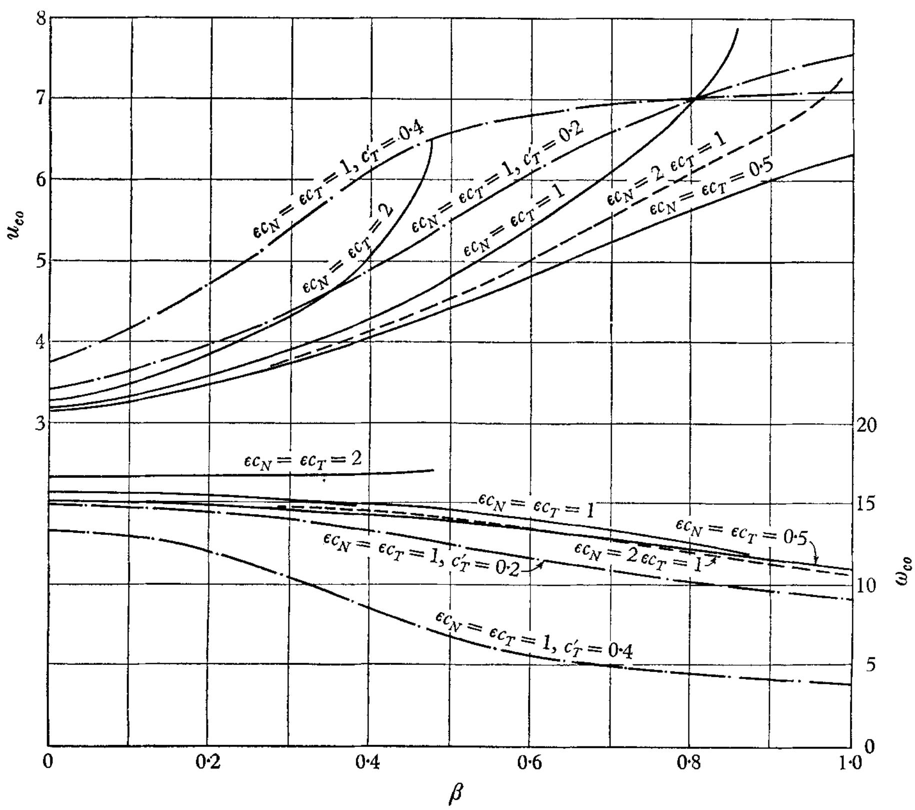

FIGURE 11. The dimensionless critical flow velocity and frequency for second-mode unstable oscillation of clamped-free cylinders ($f = 0{\cdot}8$, $\chi = 0{\cdot}01$), showing the effects of ϵc_N, ϵc_T and c_T' on stability. ———, – – – –, $c_T' = 0$; —·—, c_T' as shown.

(by determining the value of the shear force at $x = L$), and it is not surprising that it plays such a vital role on stability.

The results shown in figures 9 and 11 indicate that increasing ϵc_N and ϵc_T stabilizes the system. It is of particular interest, however, that tension due to form drag ($c'_T \neq 0$) has a destabilizing effect on buckling, as illustrated in figure 9 for $f = 0{\cdot}8$. This effect may be clarified by considering the analogous system of a taut string. The work done W in displacing the string a small distance $y(x)$ is (Morse 1948)

$$W = -\frac{1}{2}\int_0^L yT\frac{\partial^2 y}{\partial x^2}dx = -\frac{1}{2}\left[Ty\frac{\partial y}{\partial x}\right]_0^L + \frac{1}{2}\int_0^L T\left(\frac{\partial y}{\partial x}\right)^2 dx.$$

If the string, or cylinder, is supported at both ends the first term vanishes and the work done is always positive. If, however, there is a free end, the first term, which may be negative, is generally finite and may prevail over the second for certain $y(x)$. Increasing T, therefore, may have opposite effects on stability for flexible systems with one or both ends supported.

In figures 9–11 all calculations were done with $\chi = 0{\cdot}01$. It turns out that, provided it remains small, χ has a very small effect on stability, which is reasonable on physical grounds.

Some further discussion on the work presented here is included in Part 2, where the experimental work in support of this theoretical investigation is presented.

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